A Simple Approach to Mode Analysis for Parabolic Waveguides

Charles S. Kenney, *Associate Member, IEEE*, and P. L. Overfelt, *Member, IEEE*

***Abstract*—Difficulty in obtaining accurate values for parabolic cylinder functions has been an impediment to mode analysis for parabolic waveguides. A simple method, based on one-dimensional analytic continuation, is presented which gives essentially exact values for these functions; i.e., the relative error in the computed result is on the order of the machine round-off. When supplemented with a Newton–Poisson shooting method and simple homotopy techniques, this continuation method can be used to find the TE and TM mode eigenvalues, and associated separation constants, for arbitrary parabolic domains. These methods are then used to compute a power handling efficiency factor for a range of parabolic regions.**

I. Introduction

We consider parabolic cylinders of uniform cross section in the confocal parabolic coordinates (ξ, η, z), which are related to rectangular coordinates (X, Y, Z) via

$$X = \frac{1}{2}(\eta^2 - \xi^2) \qquad Y = \eta\xi \qquad z = Z \quad (1)$$

(see Fig. 1). The cross sections of interest consist of the interior regions $\Omega = \Omega(\xi_0, \eta_0)$ bounded on the right by the curve $\eta = \eta_0$ and on the left by the curve $\xi = \xi_0$.

Assuming a uniform, perfectly conducting waveguide of parabolic cross section with $e^{-i\beta z}e^{i\omega t}$ dependence, we use

$$\psi = \begin{Bmatrix} E_z \\ H_z \end{Bmatrix}$$

where ψ satisfies

$$\psi_{XX} + \psi_{YY} = -k^2\psi \qquad \text{in } \Omega \quad (2)$$

subject to the boundary conditions,

$$E_z = 0 \qquad \text{on } \partial\Omega \text{ (TM modes)} \quad (3)$$

or

$$\frac{\partial H_z}{\partial n} = 0 \qquad \text{on } \partial\Omega \text{ (TE modes)}. \quad (4)$$

Here $\partial\Omega$ denotes the boundary of Ω, $\partial/\partial n$ is the outward normal derivative, and variable subscripts indicate differentiation. In (2), $k^2 = k_0^2 - \beta^2$ and $k_0^2 = \omega^2\epsilon_0\mu_0$.

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C. S. Kenney is with the Electrical and Computer Engineering Department, University of California, Santa Barbara, CA 93106, and the Naval Weapons Center, China Lake, CA 93555.

P. L. Overfelt is with the Physics Division, Research Department, Naval Weapons Center, China Lake, CA 93555-6001.

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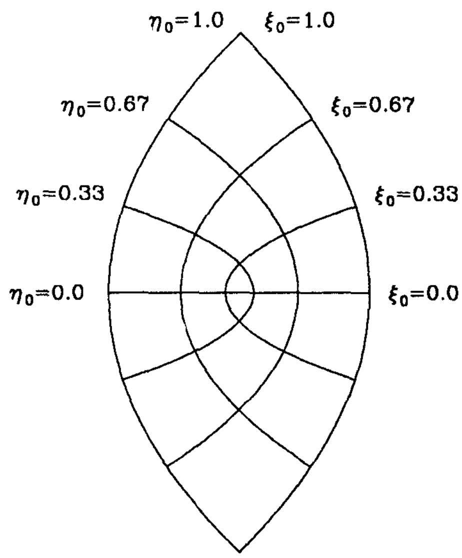

Fig. 1. Confocal parabolic regions.

Expressing (2) in parabolic coordinates gives

$$\psi_{\xi\xi} + \psi_{\eta\eta} = -k^2(\xi^2 + \eta^2)\psi. \quad (5)$$

Using separation of variables with $\psi = U(\xi)V(\eta)$ we find

$$U_{\xi\xi} + (k^2\xi^2 + \alpha)U = 0 \quad (6)$$

$$V_{\eta\eta} + (k^2\eta^2 - \alpha)V = 0 \quad (7)$$

where α is the separation constant. Equations (6) and (7) must be supplemented by the boundary conditions

$$U(\xi_0) = 0 \qquad V(\eta_0) = 0 \qquad \text{(TM modes)} \quad (8)$$

or

$$U_\xi(\xi_0) = 0, \qquad V_\eta(\eta_0) = 0 \qquad \text{(TE modes)}. \quad (9)$$

Although (6) and (7) can be solved for any α and k, the boundary conditions (8) and (9) are satisfied only at discrete pairs (α, k), when ξ_0 and η_0 are fixed.

In the following section, we show that solutions to (6) and (7) can be computed via one-dimensional analytic continuation. Section III discusses a Newton–Poisson shooting method for finding the separation constants, α, and eigenvalues, k, for fixed ξ_0 and η_0. This method is easy to use and reveals some inaccuracies in previously published work. In particular, Tables I and II give pairs of values $(a, q) = (\alpha/2k, \sqrt{2k})$ to seven significant digits for the case $\xi_0 = \eta_0$, and show that some entries in similar tables from [1] have only one digit of accuracy. The values in Tables I and II were used to generate (via a simple homotopy method) the eigenvalues of the first nine TM modes and the first eight TE modes for $\xi_0 = 1$ and $0.1 \leqslant \eta_0 \leqslant 1$, as illustrated in Figs. 6–9. These figures show that the TM modal ordering given in [2, pp. 1401–1402] is incorrect (since the TM modes immediately following k_{31}, k_{13} should be k_{40} and k_{04} rather than k_{22}). Highly accurate polynomial/rational approximations of the separation constants and eigenvalues are also given in this section for the first three TM and TE modes for $0.1 \leqslant \eta_0 \leqslant 1$ and $\xi_0 = 1$.

The last section of this paper considers selecting η_0 (for $\xi_0 = 1$) so as to optimize the power handling capability [3], [20] of the parabolic waveguide. Fig. 10 shows that there are two local maxima of the power handling efficiency factor, γ, for $0 \leqslant \eta_0 \leqslant 1$. As in [4], both of these maxima occur at η_0 values for which the second and third TE mode eigenvalues are equal. The largest of these γ values is 0.4600, which occurs at $\eta_0 = 1$, and gives a symmetrical cross section to the parabolic waveguide. This compares well with $\gamma = 0.4653$ for the 2:1 rectangle and $\gamma = 0.4698$ for an ellipse of eccentricity $e = 0.8546$.

II. Analytic Continuation

In order to treat (6)–(9) in a systematic manner, we use the normalized functions, u and v, where

$$u(\sqrt{2k}\,\xi) = U(\xi) \tag{10}$$

$$v(\sqrt{2k}\,\eta) = V(\eta). \tag{11}$$

This leads to

$$u_{xx} + \left(\frac{x^2}{4} + \frac{\alpha}{2k}\right)u = 0 \tag{12}$$

$$v_{xx} + \left(\frac{x^2}{4} - \frac{\alpha}{2k}\right)v = 0 \tag{13}$$

where $x = \sqrt{2k}\,\xi$ for (12) and $x = \sqrt{2k}\,\eta$ for (13), with

$$u(\sqrt{2k}\,\xi_0) = 0 \qquad v(\sqrt{2k}\,\eta_0) = 0 \qquad \text{(TM modes)} \tag{14}$$

$$u_x(\sqrt{2k}\,\xi_0) = 0 \qquad v_x(\sqrt{2k}\,\eta_0) = 0 \qquad \text{(TE modes)}. \tag{15}$$

Both (12) and (13) have the form

$$y_{xx} + \left(\frac{x^2}{4} + a\right)y = 0. \tag{16}$$

The solution to (16) can be expressed in terms of Whittaker functions [5], [6] or Weber functions [7]–[9] but these methods are more useful for exterior parabolic problems. Another approach [10] is to expand y in a Taylor series about $x = 0$:

$$y(x) = y_0 + y_1 x + \cdots + y_n \frac{x^n}{n!} + \cdots \tag{17}$$

where y_n is the nth derivative of y at $x = 0$. We may assume that y_0 and y_1 are given (in fact $y_0 = 1$, $y_1 = 0$ generates the even, cosinelike solution to (16) as illustrated in Fig. 2; $y_0 = 0$, $y_1 = 1$ generates the odd, sinelike solution to (16)).

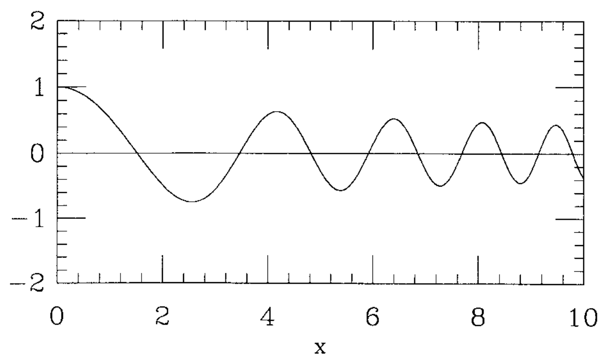

Fig. 2. Cosinelike parabolic cylinder function for $a = 1$.

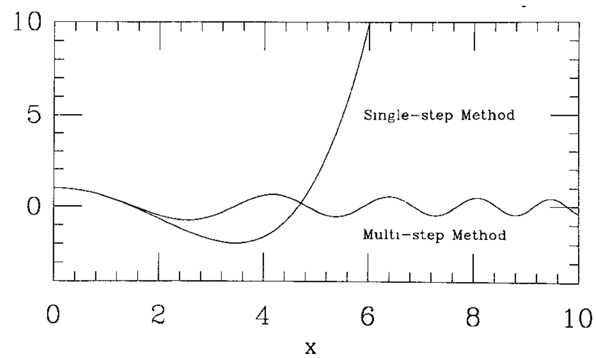

Fig. 3. Divergence of the single-step Taylor series method.

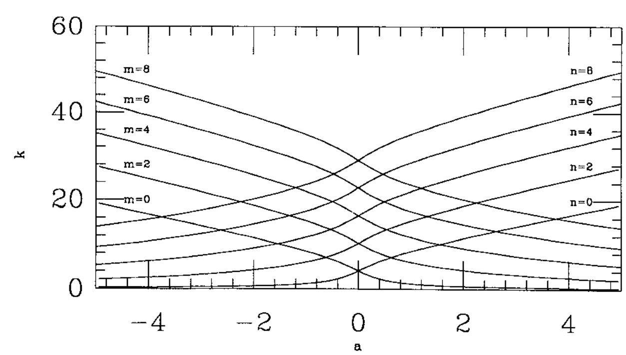

Fig. 4. Even TM mode zero curves for u(m values) and v(n values).

The higher values of y_n can be found from the recursion

$$y_n = -a y_{n-2} - \frac{1}{4}(n-2)(n-3)\,y_{n-4}. \tag{18}$$

On finite precision computers, the expansion (17) provides accurate values for $y(x)$ if x is small. However, as x increases, destructive cancellation of large positive and negative terms leads to an unnecessary loss of accuracy. For example, Fig. 3 shows that the use of (17) leads to divergence starting at about $x = 1.8$ for the even solution to (16) with $a = 1$.

This problem can be avoided by taking several small Taylor steps rather than one large step. This can be done by using the Taylor expansion of y about an arbitrary point x_0:

$$y(x_0 + h) = y_0 + y_1 h + \cdots + y_n \frac{h^n}{n!} + \cdots \tag{19}$$

where y_n now denotes the nth derivative of y at $x = x_0$. By substituting $x = x_0 + h$ into (16), the higher order values of y_n can be generated by

$$y_n = -\frac{(4a + x_0^2)}{4} y_{n-2} - \frac{(n-2)x_0}{2} y_{n-3} - \frac{(n-2)(n-3)}{4} y_{n-4}. \quad (20)$$

Differentiating (19) with respect to h gives the series expansion for y_x:

$$y_x(x_0 + h) = y_1 + y_2 h + \cdots + y_n \frac{h^{n-1}}{(n-1)!} + \cdots. \quad (21)$$

Thus to evaluate y at x, given $y(0)$ and $y_x(0)$, set $h = x/N$ for N large enough so that $|h|$ is small, say $|h| \leqslant 0.1$; then evaluate $y(h)$ and $y_x(h)$ by (19) and (21). Proceeding sequentially, evaluate y and y_x at ih for $1 \leqslant i \leqslant N$. The sums in (19) and (21) are truncated so that the pseudorelative error is less than a prescribed tolerance value ϵ. That is, pick n large enough so that

$$\frac{|y_{n+1} h^{n+1}|}{(n+1)!} \cdot \frac{1}{1 + |y(x_0)|} < \epsilon, \qquad \text{(relative error test for } y) \quad (22)$$

and

$$\frac{|y_{n+1} h^n|}{n!} \cdot \frac{1}{1 + |y_x(x_0)|} < \epsilon \qquad \text{(relative error test for } y_x). \quad (23)$$

The above method is really just a one-dimensional version of analytic continuation [11] and is sometimes referred to as a constant-step variable-order Taylor method [12]. The accuracy of the analytic continuation method was checked by using the ordinary differential equation solver LSODE [13], which employs error monitoring procedures such as those described in [14]. In all computations the analytic continuation method gave at least 11 digits of accuracy and because of its specialized nature was ten to 100 times faster than the LSODE solver.

III. Newton–Poisson Shooting Method

By combining features of the Poisson shooting method [15] with the vector form of Newton's method [12], we can solve the second-order ordinary differential equations (12) and (13) subject to (14) or (15) for fixed values of ξ_0 and η_0.

For vector valued functions, f: $R^n \to R^n$, Newton's method of solving $f(\omega) = 0$ consists of an iterative procedure:

$$\omega^{k+1} = \omega^k - (\nabla f(\omega^k))^{-1} f(\omega^k) \quad (24)$$

where $\omega^o \epsilon R^n$ is given, and $\nabla f = (\partial f_i / \partial \omega_j)$ is the gradient matrix of f.

This can be applied to the problem of solving for (a, k), such that (12)–(15) are satisfied, by setting $\omega = (a, k)$ and letting

$$f_1(a,k) = u(\sqrt{2k}\,\xi_0) \qquad f_2(a,k) = v(\sqrt{2k}\,\eta_0) \qquad \text{(TM modes)} \quad (25)$$

or

$$f_1(a,k) = u_x(\sqrt{2k}\,\xi_0) \qquad f_2(a,k) = v_x(\sqrt{2k}\,\eta_0) \qquad \text{(TE modes)} \quad (26)$$

with

$$\nabla f(a,k) = \begin{bmatrix} \partial f_1/\partial a & \partial f_1/\partial k \\ \partial f_2/\partial a & \partial f_2/\partial k \end{bmatrix}. \quad (27)$$

The values of f_1 and f_2 are easily computed by the methods of the previous section, and the derivatives in (27) can be approximated by using second-order central difference formulas. In this method, both the values ξ_0 and η_0 and the initial conditions u, u_x, v, and v_x are fixed, so that (25)–(27) do not correspond exactly to the classical Newton–Poisson shooting method [15] in which only some of the initial conditions are specified.

The iteration (24) converges quadratically to the exact solution, provided that the initial guess, $\omega^o = (a_0, k_0)$, is sufficiently close to the exact solution. It is the invertibility of the gradient matrix (see the Appendix) which accounts for this rapid convergence. This raises the question of how to select the initial values of a_0 and k_0. A lucid account of this problem is given in [2], which we paraphrase below.

Consider the problem of finding a and k for even TM modes. Let a have an arbitrary fixed value and let u and v satisfy

$$u_{xx} + \left(\frac{x^2}{4} + a\right) u = 0 \qquad u(0) = 1 \qquad u_x(0) = 0 \quad (28)$$

$$v_{xx} + \left(\frac{x^2}{4} - a\right) v = 0 \qquad v(0) = 1 \qquad v_x(0) = 0. \quad (29)$$

Then u has simple zeros $0 < z_0^+ < z_2^+ < z_4^+ \cdots$ from which we may define the values

$$k_m^+ \equiv \frac{1}{2}\left(\frac{z_m^+}{\xi_0}\right)^2. \quad (30)$$

(Even subscripts are used to indicate that u is an even function.) Similarly, v has simple zeros $0 < z_0^- < z_2^- < z_4^- \cdots$ and we set

$$k_n^- \equiv \frac{1}{2}\left(\frac{z_n^-}{\eta_0}\right)^2. \quad (31)$$

The values k_m^+ and k_n^- vary with a, and if $k_m^+ = k_n^-$ for some value $a = a_{mn}$ then for k_{mn} equal to the mutual value of k_m^+ and k_n^-, the pair (a_{mn}, k_{mn}) forces f_1 and f_2 to be zero in (25). See Fig. 4, which illustrates the intersections of these zero curves for the special case $\xi_0 = \eta_0 = 1$. Similarly, for the odd TM modes, if we let u and v satisfy (28) and (29) with the initial conditions $u(0) = 0, u_x(0) = 1, v(0) = 0, v_x(0) = 1$, then u has simple zeros $0 < z_1^+ < z_3^+ < z_5^+ \cdots$ which interlace the even zeros z_{2m}^+, and v has simple zeros $0 < z_1^- < z_3^- < z_5^- \cdots$ which interlace the even zeros z_{2n}^-. Defining k_m^+ and k_n^- as in (30) and (31), we again set $a_{mn} = a$ if $k_m^+(a) = k_n^-(a)$ (m, n both odd). This is illustrated in Fig. 5, and it should be noted that we are only interested in odd–odd or even–even intersections as discussed in [2].

Plots such as those in Figs. 4 and 5 provide approximate values for (a_{mn}, k_{mn}) which can then be refined as in

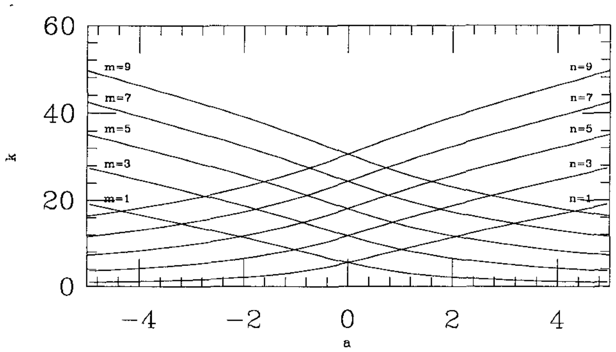

Fig. 5. Odd TM mode zero curves for u(m values) and v(n values).

(24)–(27). For the TE modes we use the same procedure but work with the zeros of u_x and v_x rather than u and v. Plots of these values are similar to the TM case.

This was done for the special case $\eta_0 = \xi_0 = 1$ and the results are given in Tables I and II for $m, n \leqslant 8$ in the form (a_{mn}, q_{mn}) where $q_{mn} = \sqrt{2k_{mn}}$. (Compare with the results in [1].)

Remark: The cumbersome method of graphically determining (a_{mn}, k_{mn}) for arbitrary $\hat{\xi}_0$ and $\hat{\eta}_0$ can be avoided by using the values in Tables I and II (or those in [1]) in the following way. Starting with (a_{mn}, k_{mn}) from the tables, move in small steps from $(\xi_0, \eta_0) = (1,1)$ to $(\hat{\xi}_0, \hat{\eta}_0)$ by using the Newton–Poisson method to find (a_{mn}, k_{mn}) at each step with the previous values used as starting values. This method is very fast and was used to generate the values of a_{mn} and k_{mn} for $m+n \leqslant 4$ (i.e., the first nine TM modes and the first eight TE modes) with $\xi_0 = 1$ and $0.1 \leqslant \eta_0 \leqslant 1$, as illustrated in Figs. 6–9. These values were also used to generate the highly accurate polynomial/rational approximations given in Tables III and IV. A comparison of the lowest TE and TM eigenvalues from various sources [1], [2], [10], [19] is given in Table V.

IV. Power Handling Capability of Parabolic Waveguides

In [3], Baum defines the efficiency factor, γ, for a planar domain Ω as

$$\gamma = \frac{k_2}{2\pi}\left(1 - \frac{k_1^2}{k_2^2}\right)^{1/4}\left[\frac{\int_\Omega |\nabla\psi_1|^2}{\max_\Omega |\nabla\psi_1|^2}\right]^{1/2} \tag{32}$$

where $0 < k_1 \leqslant k_2$ are the two lowest nonzero TE eigenvalues and ψ_1 is the TE eigenfunction corresponding to k_1.

Using the methods of the previous two sections, we investigated $\gamma = \gamma(\eta_0)$ for confocal parabolic domains, Ω, with $\xi_0 = 1$ and $0.1 \leqslant \eta_0 \leqslant 1$. (The integral on the right-hand side of (32) was evaluated numerically by using a 24-point Gaussian product formula, which is exact on polynomials up through order 47.) The results are given in Fig. 10, where we see that the overall maximum occurs at $\eta_0 = 1$ with $\gamma(1) = 0.4600$. A secondary maximum occurs at $\eta_0 \cong 0.14083$ with $\gamma(0.14083) \cong 0.4234$. Intermediate between these maxima, γ attains a minimum value of zero at $\eta_0 \simeq 0.31297$. As per the discussion in [4], these extreme values occur at crossing points for the second and third TE eigenvalues (for the maxima at $\eta_0 = 1$ and $\eta_0 = 0.14083$) or the crossing point for the first and second TE eigenvalues ($\eta_0 = 0.31297$).

We can interpret these results by comparison with the efficiency factor for a rectangle with aspect ratio r = height/width. In this case the maximum efficiency factor is $0.4653 = (3/64)^{1/4}$ which is attained at $r = 2$ and $r = 1/2$. Between these two maxima, the efficiency factor reaches a minimum of zero at $r = 1$.

For a confocal parabolic region, Ω, with $\xi_0 = 1$, the aspect ratio of the height to the width is given by

$$r = \frac{4\eta_0}{1+\eta_0^2}. \tag{33}$$

If the efficiency factor depended only on r, we would then expect γ to be maximized at $r = 2$ and $r = 1/2$ and minimized at $r = 1$ by analogy with the rectangle. That is, we would expect from (33) to see maximum efficiency at $\eta_0 = 1$ (for $r = 2$) and at $\eta_0 = 4 - \sqrt{15} = 0.12702$ (for $r = 1/2$). The corresponding minimum would then be at $\eta_0 = 2 - \sqrt{3} = 0.26795$ (for $r = 1$). Since these values are close to the true values of $\eta_0 = 1$, $\eta_0 = 0.14083$, and $\eta_0 = 0.31297$, we conclude that for parabolic waveguides the aspect ratio of the cross section is of prime importance in determining the power handling capability—just as in the case of rectangular and triangular waveguides [16], [17], [20].

V. Conclusion

Mode analysis for confocal coaxial parabolic regions is greatly simplified by using analytic continuation to evaluate parabolic cylinder functions. When combined with Newton–Poisson shooting and homotopy methods, this continuation technique easily generates the separation constants and eigenvalues of arbitrary parabolic regions, and has been applied to the problem of determining power handling capabilities for such regions.

Appendix
Nonsingularity of Newton's Method

We need two technical lemmas to show that the gradient matrix, ∇f, in (24) is nonsingular.

Lemma 1: Let y satisfy (16) with $y^2(0) + y_x^2(0) \neq 0$. Then, for any $x_0 \geqslant 0$, $y^2(x_0) + y_x^2(x_0) \neq 0$. Moreover, if $(y(0), y_x(0)) = (1,0)$ or $(0,1)$ and $y_x(x_0) = 0$, then $y_{xx}(x_0) \neq 0$.

Proof: If $y^2(x_0) + y_x^2(x_0) = 0$ for some x_0, then $y(x) = 0$ for all x by (19) and (20). This would contradict the assumption that $y^2(0) + y_x^2(0) \neq 0$. Now suppose that $y_x(x_0) = 0$. By (16) $y_{xx}(x_0) = -(x_0^2/4 + a)y(x_0)$. Since $y(x_0) \neq 0$, $y_{xx}(x_0)$ is nonzero unless $a = -x_0^2/4$. Case 1: suppose that $(y(0), y_x(0)) = (1,0)$ and that $a = -x_0^2/4$. Then $y_{xx}(x) = -(x^2/4 + a)y(x) = (x_0^2/4 - x^2/4)y(x) > 0$, and $y_x(x)$ is increasing for x near zero. Thus $y(x)$ and $y_x(x)$ increase together until $(x_0^2/4 - x^2/4)y(x)$ changes sign, which occurs at $x = x_0$. In particular, this means that at $x = x_0$, y_x is nonzero, which is a contradiction. Case 2: suppose that $(y(0), y_x(0)) = (0,1)$ and $a = -x_0^2/4$. Again, $y_{xx}(x) > 0$ for $0 < x < x_0$ and $y_x(x_0) > 0$, which leads to a contradiction. Thus in either case if $(y(0), y_x(0)) = (1,0)$ or $(0,1)$ and $y_x(x_0) = 0$, then $y_{xx}(x_0) \neq 0$.

Lemma 2: Let $k_m^+ = k_m^+(a, \xi_0)$ and $k_n^- = k_n^-(a, \eta_0)$ be defined by (30) and (31) respectively. Then $\partial k_m^+/\partial a < 0$ and $\partial k_n^-/\partial a > 0$.

TABLE I

VALUES OF THE SEPARATION CONSTANT, a_{mn} (UPPER NUMBERS), AND THE EIGENVALUE PARAMETER, $q_{mn}=\sqrt{2k_{mn}}$ (LOWER NUMBERS), FOR THE TE MODES WHERE $\xi_0=\eta_0$

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$
$n=0$			0.810270		1.937563		3.168368		4.429902
			2.701082		3.641106		4.364091		4.976563
$n=1$		0.0		1.331693		2.547107		3.796421	
		2.057677		3.213575		4.020361		4.680999	
$n=2$	−0.810270		0.0		0.572512		1.415250		2.412445
	2.701082		3.736848		4.501579		5.142995		5.704098
$n=3$		−1.331693		0.0		0.932287		1.900718	
		3.213575		4.139591		4.835016		5.431798	
$n=4$	−1.937563		−0.572512		0.0		0.513563		1.251464
	3.641106		4.501579		5.158541		5.735390		6.254941
$n=5$		−2.547107		−0.932287		0.0		0.817493	
		4.020361		4.835016		5.455439		6.001765	
$n=6$	−3.168368		−1.415250		−0.513563		0.0		0.481947
	4.364091		5.142995		5.735390		6.261505		6.744475
$n=7$		−3.796421		−1.900718		−0.817493		0.0	
		4.680999		5.431798		6.001765		6.507896	
$n=8$	−4.429902		−2.412445		−1.251464		−0.481947		0.0
	4.976563		5.704098		6.254941		6.744475		7.196398

TABLE II

VALUES OF THE SEPARATION CONSTANT, a_{mn} (UPPER NUMBERS), AND THE EIGENVALUE PARAMETER, $q_{mn}=\sqrt{2k_{mn}}$ (LOWER NUMBERS), FOR THE TM MODES WHERE $\xi_0=\eta_0$

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$
$n=0$	0.0		0.620562		1.545732		2.621655		3.756773
	2.832878		3.766949		4.501991		5.123188		5.670876
$n=1$		0.0		1.024922		2.072334		3.183781	
		3.335199		4.153264		4.823688		5.404627	
$n=2$	−0.620562		0.0		0.534856		1.312944		2.245691
	3.766949		4.526837		5.170907		5.738399		6.249205
$n=3$		−1.024922		0.0		0.859836		1.764771	
		4.153264		4.860511		5.463204		5.999933	
$n=4$	−1.5475732		−0.534856		0.0		0.494870		1.197973
	4.501991		5.170907		5.747330		6.268620		6.747111
$n=5$		−2.072334		−0.859836		0.0		0.780581	
		4.823688		5.463204		6.014112		6.512936	
$n=6$	−2.621655		−1.312944		−0.494870		0.0		0.470105
	5.123188		5.738399		6.268620		6.751731		7.201158
$n=7$		−3.183781		−1.764771		−0.780581		0.0	
		5.404627		5.999933		6.512936		6.980356	
$n=8$	−3.756773		−2.245691		−1.197973		−0.470105		0.0
	5.670876		6.249205		6.747111		7.201158		7.625300

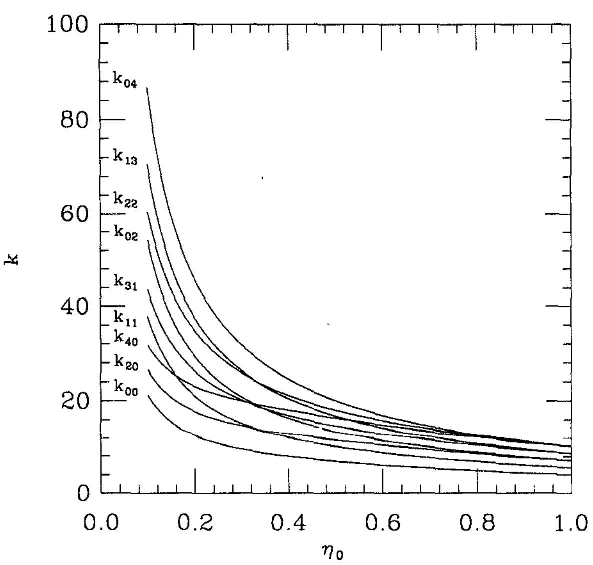

Fig. 6. TM mode eigenvalues.

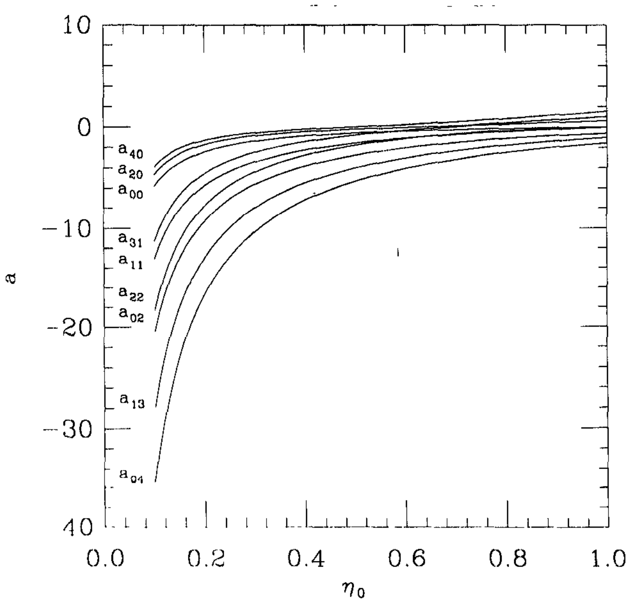

Fig. 7. TM mode separation constants.

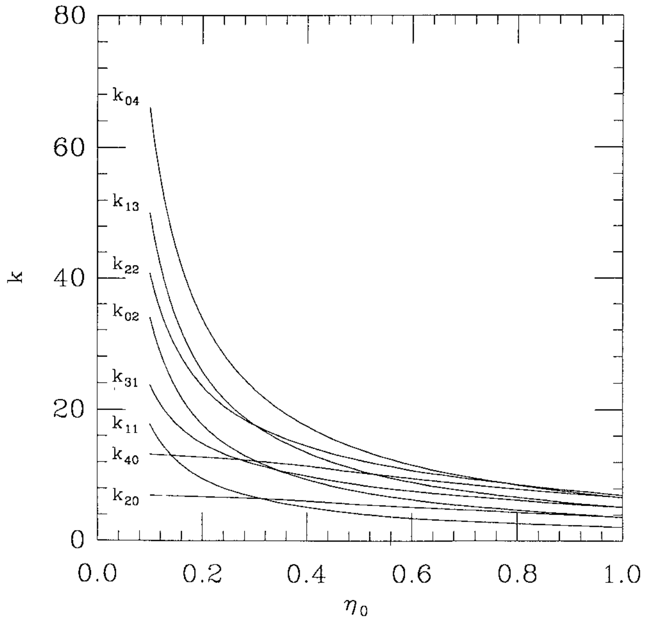

Fig. 8. TE mode eigenvalues.

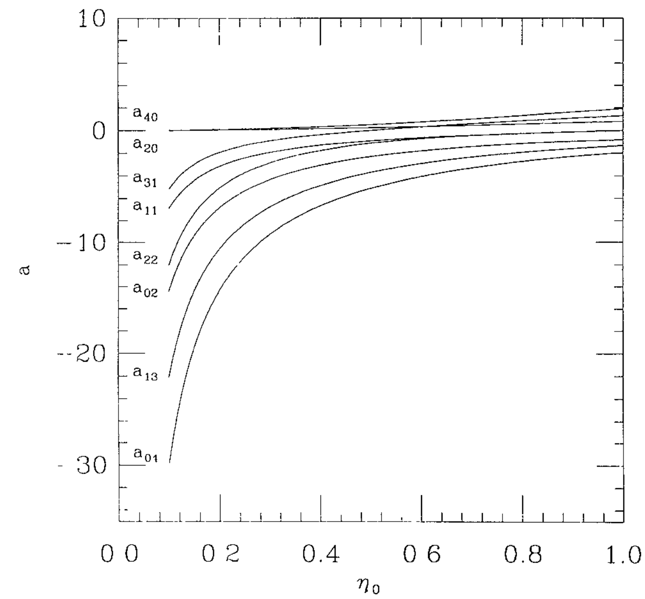

Fig. 9. TE mode separation constants.

TABLE III
EIGENVALUE AND SEPARATION CONSTANT APPROXIMATIONS FOR THE FIRST THREE TE MODES

Mode m, n	Approximation of the Separation Constant, a, and the Eigenparameter $q=\sqrt{2k}$	η_0 Interval	Maximum Relative Error (Percent)
TE 1, 1	$a=2.596929-5.094294/\eta_0+3.870710/\eta_0^2-1.651439/\eta_0^3+0.2780933/\eta_0^4$	[0.7, 1.0]	0.002
	$q=0.5312255+2.268693/\eta_0-1.038856/\eta_0^2+0.3486636/\eta_0^3-5.20495\times10^{-2}/\eta_0^4$	[0.7, 1.0]	0.0001
	$a=1.78076-2.734267/\eta_0+1.296147/\eta_0^2-0.396096/\eta_0^3+4.73622\times10^{-2}/\eta_0^4$	[0.5, 0.7]	0.0005
	$q=0.669376+1.874459/\eta_0-0.614325/\eta_0^2+0.144252/\eta_0^3-1.49289\times10^{-2}/\eta_0^4$	[0.5, 0.7]	0.00006
	$a=1.144576-1.462181/\eta_0+0.334325/\eta_0^2-7.02466\times10^{-2}/\eta_0^3+5.64386\times10^{-3}/\eta_0^4$	[0.3, 0.5]	0.003
	$q=0.849418+1.52112/\eta_0-0.352046/\eta_0^2+5.69694\times10^{-2}/\eta_0^3-3.94283\times10^{-3}/\eta_0^4$	[0.3, 0.5]	0.0007
	$a=0.689404-0.871946/\eta_0+4.44974\times10^{-2}/\eta_0^2-6.46391\times10^{-3}/\eta_0^3+3.44078\times10^{-4}/\eta_0^4$	[0.2, 0.3]	0.0003
	$q=1.164598+1.130233/\eta_0-0.168161/\eta_0^2+1.81028\times10^{-2}/\eta_0^3-8.31538\times10^{-4}/\eta_0^4$	[0.2, 0.3]	0.0003
	$a=0.517328-0.729115/\eta_0-4.40151\times10^{-4}/\eta_0^2-1.19650\times10^{-4}/\eta_0^3+5.39856\times10^{-6}/\eta_0^4$	[0.1, 0.2]	0.0006
	$q=1.626340+0.768274/\eta_0-6.03590\times10^{-2}/\eta_0^2+3.6451\times10^{-3}/\eta_0^3-9.51838\times10^{-5}/\eta_0^4$	[0.1, 0.2]	0.005
TE 2, 0	$a=-5.15875\times10^{-2}+0.370382\eta_0+0.306448\eta_0^2+0.365604\eta_0^3-0.180576\eta_0^4$	[0.7, 1.0]	0.0002
	$q=3.755891+3.67244\times10^{-2}\eta_0-2.616730\eta_0^2+2.034049\eta_0^3-0.508851\eta_0^4$	[0.7, 1.0]	0.00006
	$a=-3.40302\times10^{-2}+0.237057\eta_0+0.0661645\eta_0^2-3.814443\times10^{-2}\eta_0^3-1.31087\times10^{-2}\eta_0^4$	[0.5, 0.7]	0.00006
	$q=3.709501+0.274805\eta_0-3.073924\eta_0^2+2.423319\eta_0^3-0.632805\eta_0^4$	[0.5, 0.7]	0.00002
	$a=2.54804\times10^{-3}-4.54712\times10^{-2}\eta_0+1.484977\eta_0^2-1.1115\eta_0^3+0.51525\eta_0^4$	[0.3, 0.5]	0.00003
	$q=3.727512+0.125543\eta_0-2.606498\eta_0^2+1.768076\eta_0^3-0.286116\eta_0^4$	[0.3, 0.5]	0.00003
	$a=3.37434\times10^{-4}-9.72281\times10^{-3}\eta_0+1.269806\eta_0^2-0.538728\eta_0^3-5.43081\times10^{-2}\eta_0^4$	[0.1, 0.3]	0.0002
	$q=3.736535+8.85305\times10^{-3}\eta_0-2.028372\eta_0^2+0.46591\eta_0^3+0.838816\eta_0^4$	[0.1, 0.3]	0.00004
TE 0, 2	$a=-0.124808+0.840906/\eta_0-2.150025/\eta_0^2+0.886461/\eta_0^3-0.139533/\eta_0^4$	[0.7, 1.0]	0.00006
	$q=-0.223678+5.144419/\eta_0-3.286735/\eta_0^2+1.268121/\eta_0^3-0.201045/\eta_0^4$	[0.7, 1.0]	0.0002
	$a=0.216079-0.417672/\eta_0-0.866347/\eta_0^2+0.302681/\eta_0^3-3.96405\times10^{-2}/\eta_0^4$	[0.5, 0.7]	0.0004
	$q=0.371520+3.463166/\eta_0-1.495452/\eta_0^2+0.415015/\eta_0^3-4.78368\times10^{-2}/\eta_0^4$	[0.5, 0.7]	0.0002
	$a=0.75421-1.524952/\eta_0-6.27731\times10^{-3}/\eta_0^2+3.94451\times10^{-3}/\eta_0^3-5.13098\times10^{-4}/\eta_0^4$	[0.3, 0.5]	0.0005
	$q=0.993797+2.219658/\eta_0-0.555766/\eta_0^2+9.68244\times10^{-2}/\eta_0^3-7.11685\times10^{-3}/\eta_0^4$	[0.3, 0.5]	0.002
	$a=0.772631-1.562411/\eta_0+1.84487\times10^{-2}/\eta_0^2-2.79513\times10^{-3}/\eta_0^3+1.45653\times10^{-4}/\eta_0^4$	[0.2, 0.3]	0.00006
	$q=1.554807+1.506145/\eta_0-0.211694/\eta_0^2+2.23491\times10^{-2}/\eta_0^3-1.02011\times10^{-3}/\eta_0^4$	[0.2, 0.3]	0.0003
	$a=0.689837-1.496185/\eta_0-1.65905\times10^{-3}/\eta_0^2-4.98358\times10^{-5}/\eta_0^3+3.5604\times10^{-6}/\eta_0^4$	[0.1, 0.2]	0.0003
	$q=2.110509+1.069456/\eta_0-8.12742\times10^{-2}/\eta_0^2+4.80549\times10^{-3}/\eta_0^3-1.23783\times10^{-4}/\eta_0^4$	[0.1, 0.2]	0.005

Proof: By (28)–(31), $k_n^-(a,\eta_0)=k_m^+(-a,\eta_0)$ so we need only show that $\partial k_m^+/\partial a<0$. By definition, k_m^+ is the mth zero of u in (28). By the Sturm separation theorem [18], the zeros of u and u_x are simple and interlace each other. We now show that the distance between a zero of u and the next consecutive zero of u_x decreases with a. Similar arguments show that the distance between a zero of u_x and the next consecutive zero of u also decreases with a, thus completing the proof.

Suppose that $u(x_0)=0$ and $u_x(x_0)>0$. Let $\hat{u}(x_0)=0$ and $\hat{u}_x(x_0)=u_x(x_0)$ with $u_{xx}=-(x^2/4+a)u$, $\hat{u}_{xx}=-(x^2$

TABLE IV
EIGENVALUE AND SEPARATION CONSTANT APPROXIMATIONS FOR THE FIRST THREE TM MODES

Mode m, n	Approximation of the Separation Constant, a, and the Eigenparameter $q=\sqrt{2k}$	η_0 Interval	Maximum Relative Error (Percent)
TM 0, 0	$a=1.70632-3.557211/\eta_0+2.942827/\eta_0^2-1.319285/\eta_0^3+0.227349/\eta_0^4$	[0.7, 1.0]	0.002
	$q=4.48325\times10^{-2}+4.810751/\eta_0-2.785871/\eta_0^2+0.875188/\eta_0^3-0.112022/\eta_0^4$	[0.7, 1.0]	0.00003
	$a=1.027789-1.590098/\eta_0+0.79150/\eta_0^2-0.26778/\eta_0^3+3.36409\times10^{-2}/\eta_0^4$	[0.5, 0.7]	0.0006
	$q=0.268061+4.220208/\eta_0-2.199295/\eta_0^2+0.615888/\eta_0^3-6.89754\times10^{-2}/\eta_0^4$	[0.5, 0.7]	0.0002
	$a=0.560393-0.649577/\eta_0+7.60503\times10^{-2}/\eta_0^2-2.40106\times10^{-2}/\eta_0^3+2.26595\times10^{-3}/\eta_0^4$	[0.3, 0.5]	0.002
	$q=1.220774+2.369542/\eta_0-0.840459/\eta_0^2+0.168804/\eta_0^3-1.33536\times10^{-2}/\eta_0^4$	[0.2, 0.5]	0.003
	$a=0.388644-0.413136/\eta_0-4.63114\times10^{-2}/\eta_0^2+4.17285\times10^{-3}/\eta_0^3-1.69243\times10^{-4}/\eta_0^4$	[0.2, 0.3]	0.00004
	$q=2.312298+0.962536/\eta_0-0.153552/\eta_0^2+1.84657\times10^{-2}/\eta_0^3-9.25839\times10^{-4}/\eta_0^4$	[0.2, 0.3]	0.0004
	$a=0.487549-0.487708/\eta_0-2.50051\times10^{-2}/\eta_0^2+1.43627\times10^{-3}/\eta_0^3-3.58552\times10^{-5}/\eta_0^4$	[0.1, 0.2]	0.002
	$q=2.765350+0.588592/\eta_0-3.64749\times10^{-2}/\eta_0^2+2.0048\times10^{-3}/\eta_0^3-5.00854\times10^{-5}/\eta_0^4$	[0.1, 0.2]	0.003
TM 1, 1	$a=4.501492-9.12098/\eta_0+7.266212/\eta_0^2-3.193323/\eta_0^3+0.546599/\eta_0^4$	[0.7, 1.0]	0.003
	$q=0.395787+4.821031/\eta_0-2.600728/\eta_0^2+0.828388/\eta_0^3-0.10928/\eta_0^4$	[0.7, 1.0]	0.00003
	$a=2.86579-4.386343/\eta_0+2.096153/\eta_0^2-0.670136/\eta_0^3+8.24485\times10^{-2}/\eta_0^4$	[0.5, 0.7]	0.0008
	$q=0.630917+4.19092/\eta_0-1.965587/\eta_0^2+0.542911/\eta_0^3-6.09916\times10^{-2}/\eta_0^4$	[0.5, 0.7]	0.0002
	$a=1.752483-2.141137/\eta_0+0.384543/\eta_0^2-8.57397\times10^{-2}/\eta_0^3+7.08389\times10^{-3}/\eta_0^4$	[0.3, 0.5]	0.003
	$q=1.472579+2.549088/\eta_0-0.754941/\eta_0^2+0.14289/\eta_0^3-1.10178\times10^{-2}/\eta_0^4$	[0.3, 0.5]	0.002
	$a=1.196551-1.41065/\eta_0+2.13504\times10^{-2}/\eta_0^2-4.88571\times10^{-3}/\eta_0^3+2.94922\times10^{-4}/\eta_0^4$	[0.2, 0.3]	0.0002
	$q=2.346607+1.422418/\eta_0-0.204756/\eta_0^2+2.24217\times10^{-2}/\eta_0^3-1.05329\times10^{-3}/\eta_0^4$	[0.2, 0.3]	0.0003
	$a=1.050923-1.286222/\eta_0-1.87849\times10^{-2}/\eta_0^2+9.01076\times10^{-4}/\eta_0^3-1.94463\times10^{-5}/\eta_0^4$	[0.1, 0.2]	0.0002
	$q=2.903135+0.978869/\eta_0-7.04125\times10^{-2}/\eta_0^2+4.10207\times10^{-3}/\eta_0^3-1.05171\times10^{-4}/\eta_0^4$	[0.1, 0.2]	0.004
TM 2, 0	$a=-2.385995+8.590482\eta_0-12.73055\eta_0^2+9.811075\eta_0^3-2.664454\eta_0^4$	[0.7, 1.0]	0.0005
	$q=6.093106-2.599687\eta_0-9.91782\times10^{-2}\eta_0^2+0.457729\eta_0^3-8.50192\times10^{-2}\eta_0^4$	[0.7, 1.0]	0.0002
	$a=-3.199621+13.4697\eta_0-23.74494\eta_0^2+20.89869\eta_0^3-6.861979\eta_0^4$	[0.5, 0.7]	0.002
	$q=6.843306-6.925676\eta_0+9.312155\eta_0^2-8.695138\eta_0^3+3.271385\eta_0^4$	[0.5, 0.7]	0.0002
	$a=-5.356343+31.04948\eta_0-77.79276\eta_0^2+95.15671\eta_0^3-45.31746\eta_0^4$	[0.35, 0.5]	0.003
	$q=7.952511-15.92431\eta_0+36.85081\eta_0^2-46.36436\eta_0^3+22.69698\eta_0^4$	[0.35, 0.5]	0.0003
	$a=-8.586632+68.25198\eta_0-239.4341\eta_0^2+409.1163\eta_0^3-275.2575\eta_0^4$	[0.25, 0.35]	0.002
	$q=9.274474-31.02869\eta_0+101.96667\eta_0^2-171.8812\eta_0^3+113.9531\eta_0^4$	[0.25, 0.35]	0.0003
	$a=-13.35043+145.7653\eta_0-715.2023\eta_0^2+1714.301\eta_0^3-1625.025\eta_0^4$	[0.175, 0.25]	0.003
	$q=10.70829-54.1884\eta_0+243.1092\eta_0^2-556.4399\eta_0^3+509.0721\eta_0^4$	[0.175, 0.25]	0.0003
	$a=-23.22342+386.3231\eta_0-2932.606\eta_0^2+10867.95\eta_0^3-15885.19\eta_0^4$	[0.1, 0.175]	0.03
	$q=13.04974-110.8061\eta_0+761.2537\eta_0^2-2681.125\eta_0^3+3798.857\eta_0^4$	[0.1, 0.175]	0.004

TABLE V
COMPARISON OF EIGENPARAMETERS, $q_{mn}=\sqrt{2k_{mn}}$, AND SEPARATION CONSTANTS a_{mn} FOR $\xi_0=\eta_0=1$ FROM VARIOUS SOURCES

Mode m, n	Kenney–Overfelt		Zagrodzinski		Morse–Feshbach		Spence–Wells		Larsen	
	a_{mn}	q_{mn}	a_{mn}	q_{mn}	a_{mn}	q_{mn}	a_{mn}	q_{mn}	a_{mn}	q_{mn}
TM 0, 0	0.0	2.832878	0.0	2.83	0.0	2.8327	0.0	2.833	0.0	2.85
TM 1, 1	0.0	3.335199	0.0	3.34	0.0	3.3353	0.0	3.335	0.0	3.35
TM 2, 0	0.620562	3.766949	0.58	3.77	–	–	–	–	0.62	3.78
TM 3, 1	1.024922	4.153264	1.02	4.16	–	–	–	–	1.02	4.16
TM 4, 0	1.545732	4.501991	1.54	4.49	–	–	–	–	–	–
TM 2, 2	0.0	4.526837	0.0	4.53	0.0	4.5268	0.0	4.527	0.0	4.50
TE 1, 1	0.0	2.057677	0.0	2.06	–	–	0.0	2.061	0.0	2.08
TE 2, 0	0.810270	2.701082	0.81	2.71	–	–	–	–	0.81	2.72
TE 3, 1	1.331693	3.213575	1.33	3.21	–	–	–	–	1.33	3.22
TE 4, 0	1.937563	3.641106	1.94	3.64	–	–	–	–	–	–
TE 2, 2	0.0	3.736848	0.0	3.74	–	–	0.0	3.737	0.0	3.73
TE 5, 1	2.547107	4.020361	2.54	4.02	–	–	–	–	–	–
TE 3,3	0.0	4.139591	0.0	4.14	–	–	0.0	4.147	0.0	4.14

$/4+a+\epsilon)\hat{u}$, for $\epsilon>0$. Let x_1 and $\hat{x}_1$ denote respectively the first zero larger than x_0 of u_x and $\hat{u}_x$. Now define $\omega=\hat{u}_x=u_x$, then $\omega_x=-(x^2/4+a)\omega-\epsilon\hat{u}$.

As in Lemma 1, we may assume that $x_0^2/4+a>0$. At $x=x_0$, $\omega=0$, $\omega_x=0$, and $\omega_{xx}=-\epsilon\hat{u}_x(x_0)<0$. Thus $\omega(x_0+\delta)<0$ for δ small and positive. Now suppose that $\omega(x_2)=0$ for $x_0<x_2<\hat{x}_1$. Then $\omega_x(x_2)=-\epsilon\hat{u}(x_2)<0$ since $\hat{u}$ is positive on $(x_0,\hat{x}_1)$. This implies that $\omega\leqslant0$ over $[x_0,\hat{x}_1]$ and hence $0\leqslant\hat{u}_x<u_x$ over the same interval. That is, the distance between the zeros of u and u_x decreases as a increases. Similar arguments hold for $u(x_0)=0$ and $u_x(x_0)<0$.

Remark: Essentially the same proof shows that $\partial k_m^+/\partial a<0$ and $\partial k_n^-/\partial a>0$ for the TE modes.

Theorem: The gradient matrix, ∇f, is nonsingular at $(a,k)=(a_{mn},k_{mn})$ for $m\geqslant0, n\geqslant0$.

Proof: First consider the TM mode case: $f_1(a,k)=u(\sqrt{2k}\,\xi_0,a)$, $f_2(a,k)=v(\sqrt{2k}\,\eta_0,a)$, where the second argument denotes the dependence on a. The gradient matrix is nonsingular if $\det\nabla f\equiv(\partial f_1/\partial a)(\partial f_2/\partial k)-(\partial f_1/\partial k)(\partial f_2/\partial a)\neq0$. By definition, $u(\sqrt{2k_m^+(a)}\,\xi_0,a)=0$, $v(\sqrt{2k_n^-(a)}\,\eta_0,a)=0$. Differentiating with respect to a gives

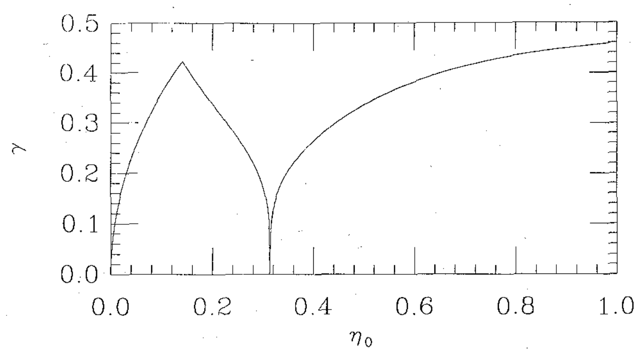

Fig. 10. Parabolic waveguide efficiency factors.

$\partial f_1/\partial a = u_a = u_x(\xi_0/2\sqrt{2k_m^+})(dk_m^+/da)$, $\partial f_2/\partial a = v_a = v_x(\eta_0/2\sqrt{2k_n^-})(dk_n^-/da)$. Also $\partial f_1/\partial k = u_x\xi_0/2\sqrt{2k}$, $\partial f_2/\partial k = v_x\eta_0/2\sqrt{2k}$. Therefore, at $a = a_{mn}$ and $k_m^+ = k_n^- = k_{mn}$, $\det \nabla f = (\xi_0\eta_0 u_x v_x/8k_{mn})(dk_n^-/da - dk_m^+/da) \neq 0$ since u_x, v_x, and $(dk_n^-/da - dk_m^+/da)$ are all nonzero by Lemmas 1 and 2.

For the TE modes we find $\det \nabla f = (\xi_0\eta_0 u_{xx} v_{xx}/8k_{mn})(dk_n^-/da - dk_m^+/da) \neq 0$, again by Lemmas 1 and 2.

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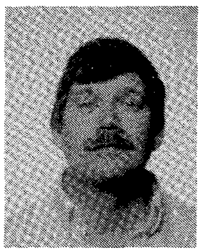

Charles S. Kenney (A'87) was born in Washington, DC, in 1950. He received the B.S., M.A., and Ph.D. degrees in mathematics from the University of Maryland, College Park, in 1973, 1976, and 1979, respectively.

From 1979 to 1981 he taught mathematics at California State College, Bakersfield, and subsequently worked as a Numerical Analyst at the Naval Weapons Center, China Lake, CA. Since 1987 he has worked half-time at the Weapons Center and half-time as an Associate Research Engineer in the Electrical and Computer Engineering Department at the University of California, Santa Barbara.

Dr. Kenney is a member of SIAM with research interests in numerical linear algebra, control theory, and partial differential equations.

P. L. Overfelt (M'85), photograph and biography not available at the time of publication.